INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 27, 241–258 (1998)

ON A MULTILEVEL APPROACH FOR THE TWO DIMENSIONAL NAVIER–STOKES EQUATIONS WITH FINITE ELEMENTS

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SUMMARY

We study if the multilevel algorithm introduced in Debussche *et al.* (*Theor. Comput. Fluid Dynam.*, **7**, 279–315 (1995)) and Dubois *et al.* (*J. Sci. Comp.*, **8**, 167–194 (1993)) for the 2D Navier–Stokes equations with periodic boundary conditions and spectral discretization can be generalized to more general boundary conditions and to finite elements. We first show that a direct generalization, as in Calgaro *et al.* (*Appl. Numer. Math.*, **21**, 1–40 (1997)), for the Burgers equation, would not be very efficient. We then propose a new approach where the domain of integration is decomposed in subdomains. This enables us to define localized small-scale components and we show that, in this context, there is a good separation of scales. We conclude that all the ingredients necessary for the implementation of the multilevel algorithm are present. © 1998 John Wiley & Sons, Ltd.

Int. J. Numer. Meth. Fluids, **27**: 241–258 (1998)

KEY WORDS: multilevel algorithm; 2D Navier–Stokes equations; finite element; large eddy simulations; long time integration

1. INTRODUCTION

The integration of evolutionary Navier–Stokes equations for a high Reynolds number remains a difficult problem despite the significant increase in computing power. It is well known that for high Reynolds numbers, these equations have a very large number of degrees of freedom (see Reference 1 for an estimate based on physical arguments and Reference 2 for a mathematical point of view).

A classical way to avoid this problem is to average the equations and to model the resulting Reynolds stress tensor. The well-known k-ε model of turbulence for instance is obtained in this way. Similar ideas are used to derive models for large eddy simulation, such as the subgrid models. In all these theories, physical arguments are used in a crucial way when modelling the averaged non-linear term.

This work is part of a programme whose aim is to integrate Navier–Stokes equations for high Reynolds number. As in turbulence models, the idea is to simulate only the evolution of the large scales. The effect of the small scales cannot be neglected and the way it is taken into account is based on mathematical arguments.

In previous works[3,4] a multilevel scheme was proposed for the two dimensional Navier–Stokes equations on a periodic rectangle (Fourier discretization was used). In this context, once a cut-off wavelength is given, the unknown u splits naturally into two components; y representing the large scales and z representing the small scales. The equation satisfied by y contains terms depending on z,

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CCC 0271–2091/98/010241–18 $17.50 *Received May 1996*

the so-called interaction terms that play the same role as the Reynolds stress tensor. Based on a detailed analysis of the behaviour of these terms on a turbulent solution computed by direct simulation, it was derived that, although they are not negligible, the interaction terms vary slowly and their variations over sufficiently long intervals of time could be neglected. An algorithm was developed in which these terms were frozen and a totally self-adaptative procedure was obtained.

However, because Fourier discretization was used, these works are restricted to homogeneous turbulence. When more general boundary conditions are considered, finite elements can be used and it seems natural to use the hierarchical decomposition of the finite element space to define the small and large scale components. Theoretical and numerical studies[5–7] have indicated that, indeed, with this definition, the scales do separate, and, as expected, the interaction terms are small but not negligible.

Recently,[8] the idea used for the simulation of homogeneous turbulence to freeze the interaction terms has been successfully implemented in the case of a finite element discretization of the two dimensional Burgers equations; hierarchical finite elements were used.

In this work, we study how this strategy can be applied to the two-dimensional Navier–Stokes equations with general boundary conditions. Several questions have to be answered before implementing a multilevel algorithm as in the above-mentioned work. Do the interaction terms vary slowly in this case? What is the effect of the pressure? In References 3, 4 or 8, only the velocity appears in the equations. Also in Reference 8, the practical estimate of the time during which the interaction terms are frozen was not totally satisfactory. It depends on the variations of these terms, which is very expensive to compute. The trick is to find another simple term that behaves similarly and whose variations are correlated to the variations of the interaction terms. A very good correlation was found in the periodic case when the time derivatives of the small scale components was used. However, the correlation was rather poor in Reference 8.

We first present the basic numerical method used to solve the Navier–Stokes equations. To avoid problems with the pressure we have considered a penalty method, offering the possibility of eliminating it. A mixed formulation using the $4P1 - P1$—also called $P1$ iso $P2 - P1$—element is chosen. This is motivated by the possibility of easily defining a hierarchical basis and the ability of this element to compute non-stationary flows.

We have chosen the regularized driven cavity at Reynolds number 5000 as our test problem and we show that the method gives good results.

In section 3, we recall the definition of hierarchical basis and write the equations satisfied by the small and large scale components. Section 4 contains our analysis. First, we generalize the algorithm used in References 4 and 8 so that it can be implemented in our case. Then, we give a relation between the time during which the interaction terms can be frozen and their variations. Thus, by taking the values of these variations for the solution computed in section 1, we are able to give estimates of this time. It appears that, even though the penalty method is used, the pressure has a rather bad effect. The interaction terms containing small-scale components of the pressure cannot be frozen for more than one or two time steps.

To overcome this problem, we propose a new approach in which the local nature of the finite element discretization is used and local small scale components are considered. The domain of computation is decomposed in subdomains providing a natural definition of localized small scales. We show that there is a very good separation of scales and that the interaction terms can be frozen on long intervals of time but locally. Also we have been able to exhibit a very good correlation between the variations of the different terms.

It is easy to generalize the multilevel algorithm so that it includes the notion of local small scales. Moreover, this can be coupled with parallel computation techniques. Thus, it seems that a very efficient procedure can be derived.

In a future work, we will implement such an algorithm coupled with domain decomposition and parallel computation. Also we will consider more physical flows and higher Reynolds numbers. We think that the conclusion of this work will still be valid.

2. THE EQUATIONS AND THEIR DISCRETE FORMULATION

2.1. The continuous problem

We consider the motion of a two dimensional incompressible viscous fluid in a domain Ω. In non-dimensional form, the equations satisfied by the velocity $u(x, t)$ and the pressure $p(x, t)$, $x \in \Omega$, $t > 0$ are

$$\frac{\partial u}{\partial t} - \frac{1}{Re}\Delta u + (u \cdot \nabla)u + \nabla p = f, \qquad \nabla \cdot u = 0, \tag{1}$$

in $\Omega \times \mathrm{R}^+$. Here Re denotes the Reynolds number:

$$Re = UL/v,$$

U has being a characteristic velocity of the flow, L a characteristic length and v the kinematic viscosity. Also, f represents the external forcing. Equations (1) are supplemented with an initial condition

$$u(x, 0) = u_0(x), \qquad x \in \Omega.$$

Our test problem is the regularized driven cavity, thus Ω is the unit square $[0, 1]^2$ and the boundary conditions are

$$u(x, t) = g(x), \qquad x \in \partial\Omega, \qquad t > 0, \tag{2}$$

where g is 0 on $\partial\Omega$ except when $x_2 = 1$ and

$$g(x_1, 1) = \begin{pmatrix} (1 - (2x_1 - 1)^2)^\alpha \\ 0 \end{pmatrix}, \qquad x_1 \in [0, 1].$$

To avoid the problem due to the incompressibility condition, we use the penalty method.[9,10] We introduce a small parameter $\varepsilon > 0$ and approximate (1) by the following system

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \frac{1}{Re}\Delta u_\varepsilon + (u_\varepsilon \cdot \nabla)u_\varepsilon + \tfrac{1}{2}(\nabla \cdot u_\varepsilon)u_\varepsilon + \nabla p_\varepsilon &= f, \\ \nabla \cdot u_\varepsilon + \varepsilon p_\varepsilon &= 0, \end{aligned} \tag{3}$$

in $\Omega \times \mathrm{R}^+$. The initial and boundary conditions are unchanged. The term $\frac{1}{2}(\nabla \cdot u_\varepsilon)u_\varepsilon$ has been added so that (2) defines a well-posed problem, it does not change the qualitative behaviour of the solution. It has been shown[10,11] that the following estimate holds:

$$|u_\varepsilon - u|_{H^1(\Omega)} + |p_\varepsilon - p|_{L^2(\Omega)} \leqslant K\varepsilon,$$

where K is a positive constant. Finally, we introduce the variational formulation of (3):

Find $(u_\varepsilon(t), p_\varepsilon(t)) \in (H_g^1(\Omega))^2 \times L^2(\Omega)$ such that

$$\left(\frac{\partial u_\varepsilon}{\partial t}, v\right) + \frac{1}{Re}((u_\varepsilon, v)) + b(u_\varepsilon, u_\varepsilon, v) - (p_\varepsilon, \nabla \cdot u_\varepsilon) = (f, v),$$
$$(\nabla \cdot u_\varepsilon, q) + \varepsilon(p_\varepsilon, q) = 0 \tag{4}$$

for any $(v, q) \in (H_0^1(\Omega))^2 \times L^2(\Omega)$,

where $(\cdot, \cdot)$ is the usual $L^2(\Omega)$ inner product, $((\cdot, \cdot))$ is the bilinear form associated to the Laplace operator:

$$((u, v)) = (\nabla u, \nabla v), \qquad u, v \in H^1(\Omega),$$

b is the trilinear form defined by

$$b(u, v, w) = ((u \cdot \nabla)v, w) + \tfrac{1}{2}((\nabla \cdot u)v, w), \qquad u, v, w \in H^1(\Omega),$$

and $H_g^1(\Omega)$ (resp. $H_0^1(\Omega)$) is the subspace of $H^1(\Omega)$ consisting of functions satisfying (2) (resp. vanishing on $\partial\Omega$). We will use also the notation $|\cdot|$ for the $L^2(\Omega)$ norm and $\|\cdot\|$ for the norm associated to $((\cdot, \cdot))$. From now on, ε is a fixed parameter and we omit to write the index ε for the velocity and pressure.

2.2. *The discrete equations*

We use a finite element discretization of our problem. As is well known, it is necessary to approximate the velocity and the pressure by a couple of spaces of finite elements satisfying the Babuska–Brezzi "Inf–Sup" condition. Elements with a piecewise constant pressure (e.g. the $P2 - P0$ element where the velocity is a second degree polynomial on each triangle or the $4P1 - P0$ element where it is piecewise linear on a finer triangulation) have been widely used (see Reference 7 and its references). They are easy to implement and lead to very sparse matrix. However, it seems that they are well suited for computations of stationary flow but, when they are used for the evolutionary Navier–Stokes, oscillations appear. A nice alternative, which overcomes this problem, is to choose a discontinuous piecewise linear element for the pressure and bubble type elements for the velocity. We cannot choose such an element, because our aim is to use a hierarchical decomposition of the computed solution and hierarchical formulations are not simple to write in this case. We have chosen the $4P1 - P1$ (or $P1$ iso $P2 - P1$) element where the pressure is piecewise linear and the velocity is also piecewise linear but on a finer grid. This element satisfies the "Inf–Sup" condition.[12]

To describe more precisely the element, and to set some notations used in the next section, we introduce a family of nested meshes

$$\mathscr{T}_0 \subset \mathscr{T}_1 \subset \cdots \subset \mathscr{T}_d,$$

constructed from an initial finite element mesh $\mathscr{T}_0$ by subdividing each triangle into four similar subtriangles (see Figure 1 where three nested meshes ($d = 2$) are represented). For $k = 0, \ldots d$, V_k is the finite element space of piecewise linear functions associated to $\mathscr{T}_k$. On a given level $k \geqslant 1$, the velocity is sought in $\mathscr{V}_{g,k}$ the subspace of $V_k \times V_k$ of functions satisfying the discrete equivalent of (4), the test functions are taken in $\mathscr{V}_{0,k}$ the subspace of $V_k \times V_k$ of functions vanishing on $\partial\Omega$.

The computed pressure is taken in $\mathscr{Q}_k = V_{k-1}$, just as the test functions for the penalized incompressibility condition. We obtain the formulation:

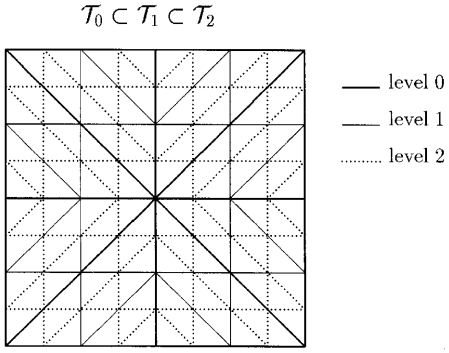

Figure 1. Hierarchical triangulation of domain Ω: $\mathcal{T}_0 \subset \mathcal{T}_1 \subset \mathcal{T}_2$

Find $(u_k(t), p_k(t))$ in $\mathcal{V}_{g,k} \times \mathcal{Q}_k$ such that

$$\left(\frac{\partial u_k}{\partial t}, v_k\right) + \frac{1}{Re}((u_k, v_k)) + b(u_k, u_k, v_k) - (p_k, \nabla \cdot v_k) = (f, v_k),$$
$$(\nabla \cdot u_k, q_k) + \varepsilon(p_k, q_k) = 0, \tag{5}$$

for any (v_k, q_k) in $\mathcal{V}_{0,k} \times \mathcal{Q}_k$.

It only remains to discretize in time, and we have used an Euler scheme; the linear part is treated implicitly and the non-linear part explicitly. Higher order schemes could be used.

It is convenient to write (5) in matrix form. Let Σ_k be the set of nodes of $\mathcal{T}_k$ and

$$n_k = \text{card } \Sigma_k, \qquad n_{0,k} = \text{card}(\Sigma_k \cap \mathring{\Omega}).$$

The canonical—or nodal—basis of V_k (resp. $V_{0,k}$) consists of the functions $\Phi_{k,i}$, $i = 1, \ldots, n_k$ (resp. $i = 1, \ldots, n_{0,k}$) whose value is 1 at the node i and 0 at the others. The canonical basis of $Q_k = V_{k-1}$ is $(\xi_{k,i})_{i=1,\ldots,m_k}$ with $m_k = n_{k-1}$ and $\xi_{k,i} = \Phi_{k-1,i}$. We define the nodal matrices

$$(\overline{\mathscr{A}}_k)_{i,j} = \frac{1}{Re}((\Phi_{k,i}, \Phi_{k,j})), \qquad 1 \leqslant i, j \leqslant n_{0,k}$$
$$(\overline{\mathscr{M}}_k)_{i,j} = \frac{1}{\Delta t}(\Phi_{k,i}, \Phi_{k,j}), \qquad 1 \leqslant i, j \leqslant n_{0,k}$$
$$(\mathscr{B}_{l,k})_{i,j} = -\left(\xi_{k,i}, \frac{\partial \Phi_{k,j}}{\partial x_l}\right), \qquad 1 \leqslant i \leqslant m_k, \quad 1 \leqslant j \leqslant n_{0,k}, \quad l = 1, 2.$$
$$(\mathscr{C}_k)_{i,j} = (\xi_{k,i}, \xi_{kj}), \qquad 1 \leqslant i, j \leqslant m_k$$

and the block matrices

$$\mathscr{A}_k = \begin{pmatrix} \overline{\mathscr{A}}_k & 0 \\ 0 & \overline{\mathscr{A}}_k \end{pmatrix}, \qquad \mathscr{M}_k = \begin{pmatrix} \overline{\mathscr{M}}_k & 0 \\ 0 & \overline{\mathscr{M}}_k \end{pmatrix}, \qquad \mathscr{B}_k = (\mathscr{B}_{1,k}, \mathscr{B}_{2,k})$$

and the vectors

$$U_k^{n+1} = \begin{pmatrix} U_{1,k}^{n+1} \\ U_{2,k}^{n+1} \end{pmatrix}, \qquad F_k = \begin{pmatrix} F_{1,k} \\ F_{2,k} \end{pmatrix}, \qquad b_k(U_k^n) = \begin{pmatrix} b_{1,k}(U_k^n) \\ b_{2,k}(U_k^n) \end{pmatrix}$$

of $\mathscr{V}_{0,k}$ associated to the new velocity, the forcing term and the explicit non-linear term and the vector P_k^{n+1} in Q_k associated to the pressure. Then we have the matrix problem

$$\begin{pmatrix} \mathscr{M}_k + \mathscr{A}_k & \mathscr{B}_k^T \\ \mathscr{B}_k & -\varepsilon \mathscr{C}_k \end{pmatrix} \begin{pmatrix} U_k^{n+1} \\ P_k^{n+1} \end{pmatrix} = \begin{pmatrix} F_k + \mathscr{M}_k U_k^n - b_k(U_k^n) \\ 0 \end{pmatrix}. \tag{6}$$

The variable P_k^{n+1} can be eliminated, obtaining

$$\left(\mathscr{M}_k + \mathscr{A}_k + \frac{1}{\varepsilon} \mathscr{B}_k^T \mathscr{C}_k^{-1} \mathscr{B}_k \right) U_k^{n+1} = F_k - b_k(U_k^n) + \mathscr{M}_k U_k^n. \tag{7}$$

We avoid the computation of $\mathscr{C}_k^{-1}$ by replacing $\mathscr{C}_k$ by a diagonal matrix obtained thanks to a mass-lumping technique.

The matrix on the left-hand side of (7) is very ill-conditioned and iterative methods are not well suited for the resolution of (7), so we have used a direct solver. Below we present the result of our

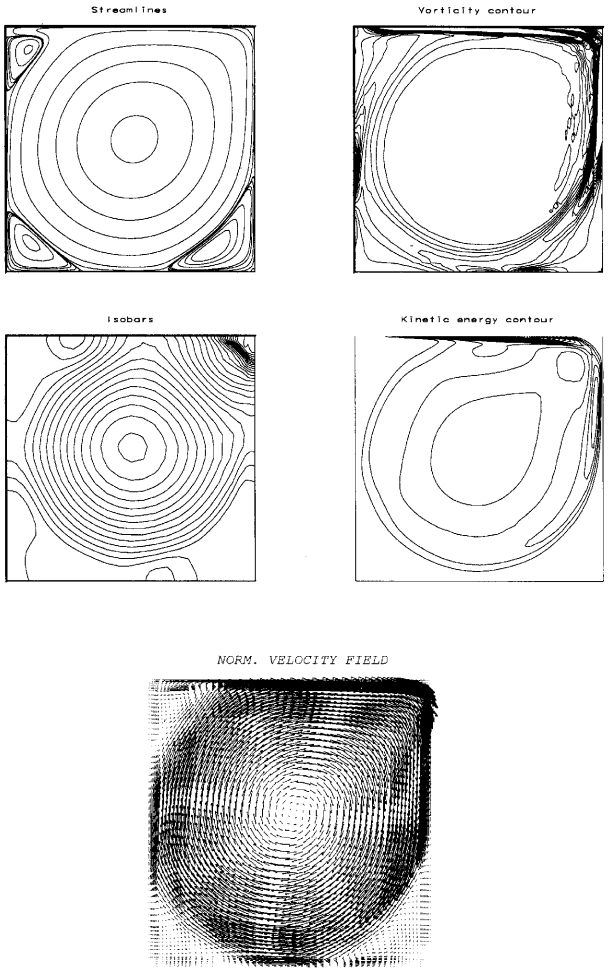

Figure 2. Regularized driven cavity at $Re = 5000$

computations for our test problem. The Reynolds number is 5000 and the parameter ε is 10^{-3}. Also, we have taken $\Delta t = 2 \times 10^{-3}$, $n_k = 65^2$. As shown in Figure 2, it gives good results.

3. THE HIERARCHICAL DECOMPOSITION

To split the solution into its small and large scale components, we use the hierarchical finite element basis. This concept has been introduced by Zienciewicz *et al.*[13] and has been developed by Yserentant[14] for elliptic linear problems. It has been shown to be useful for multilevel schemes on a previous work on the Burgers equation.[8] Let $k \geqslant 1$, obviously we have

$$V_{k-1} \subset V_k,$$

we split

$$V_k = V_{k-1} + W_k$$

and recursively

$$V_d = V_0 + W_1 + \cdots + W_d.$$

The hierarchical basis of W_k consists of the $\psi_{k,j} = \Phi_{k,j}$ where $j = n_{k-1} + 1, \ldots, n_k$ is a nodal point of $\mathscr{T}_k \backslash \mathscr{T}_{k-1}$. On V_{k-1} we take the nodal basis at level $k-1$:

$$\psi_{k,j} = \Phi_{k-1,j}, \qquad j = 1, \ldots, n_{k-1}.$$

If u_k is in V_k, its decomposition is

$$u_k = y_{k-1} + z_k, \qquad y_{k-1} \in V_{k-1}, \qquad z_k \in W_k$$

and we have

$$y_{k-1} = \sum_{i=1}^{n_{k-1}} y_{k-1,i} \psi_{k,i}, \qquad z_k = \sum_{i=n_{k-1}+1}^{n_k} z_{k,i} \psi_{k,i},$$

with

$$\begin{aligned} y_{k-1,i} &= u_k(A_i), \qquad A_i \in \Sigma_{k-1}, \\ z_{k,i} &= u_k(B_i) - \tfrac{1}{2}(u_k(A_{i1}) + u_k(A_{i2})), \qquad B_i \in \Sigma_k \backslash \Sigma_{k-1}, \end{aligned}$$

where B_i is the midpoint of the edge $[A_{i1}, A_{i2}]$. From Taylor's formula, we expect z_k to be small compared with y_{k-1}. However, and it will be made clear in the next section, this is not always true, especially if the solution has strong gradient. With obvious notation, we define a decomposition of $\mathscr{V}_{g,k}$

$$\mathscr{V}_{g,k} = \mathscr{V}_{g,k-1} + \mathscr{W}_{g,k}$$

and of $\mathscr{V}_k$, $\mathscr{V}_{0,k}$. The space $\mathscr{Q}_k$ is split into

$$\mathscr{Q}_k = \mathscr{Q}_{k-1} + \mathscr{R}_k$$

and the unknowns are written as

$$u_k = y_{k-1} + z_k, \qquad p_k = p_{k-1} + \pi_k.$$

Then the hierarchical formulation of (5) is

Find (y_{k-1}, p_{k-1}) in $\mathscr{V}_{g,k-1} \times \mathscr{Q}_{k-1}$ and (z_k, π_k) in $\mathscr{W}_{g,k} \times \mathscr{R}_k$ such that

$$\begin{cases} \left(\dfrac{\partial y_{k-1}}{\partial t}, \tilde{y}_{k-1}\right) + \dfrac{1}{Re}((y_{k-1}, \tilde{y}_{k-1})) + b(y_{k-1}, y_{k-1}, \tilde{y}_{k-1}) \\ \qquad -(p_{k-1}, \operatorname{div} \tilde{y}_{k-1}) + L(z_k, \pi_k, \tilde{y}_{k-1}) + b_{\text{int}}(y_{k-1}, z_k, \tilde{y}_{k-1}) = (f, \tilde{y}_{k-1}), \\ (\operatorname{div} y_{k-1} + \operatorname{div} z_k, \tilde{p}_{k-1}) + \varepsilon(p_{k-1} + \pi_k, \tilde{p}_{k-1}) = 0 \end{cases} \tag{8}$$

for any $(\tilde{y}_{k-1}, \tilde{p}_{k-1})$ in $\mathscr{V}_{0,k-1} \times \mathscr{Q}_{k-1}$. And

$$\begin{cases} \left(\dfrac{\partial z_k}{\partial t}, \tilde{z}_k\right) + \dfrac{1}{Re}((z_k, \tilde{z}_k)) + b(z_k, z_k, \tilde{z}_k) \\ \qquad -(\pi_k, \operatorname{div} \tilde{z}_k) + L(y_{k-1}, p_{k-1}, \tilde{z}_k) + b_{\text{int}}(z_k, y_{k-1}, \tilde{z}_k) = (f, \tilde{z}_k), \\ (\operatorname{div} y_{k-1} + \operatorname{div} z_k, \tilde{\pi}_k) + \varepsilon(p_{k-1} + \pi_k, \tilde{\pi}_k) = 0, \end{cases} \tag{9}$$

for any $(\tilde{z}_k, \tilde{\pi}_k) \in \mathscr{W}_{0,k} \times \mathscr{R}_k$.

The function b_{int} represents the non-linear interaction between the large and small scale components, it is defined by

$$b_{\text{int}}(u, v, w) = b(u, v, w) + b(v, u, w) + b(v, v, w)$$

and L is the linear interaction term

$$L(u, p, w) = \left(\frac{\partial u}{\partial t}, w\right) + \frac{1}{Re}((u, w)) - (p, \operatorname{div} w).$$

More generally, the unknown could be split according to the sum $V_k = V_0 + W_1 + \cdots + W_k$ as

$$\begin{aligned} u_k &= y_0 + z_1 + \cdots + z_k, \\ p_k &= p_0 + \pi_1 + \cdots + \pi_k. \end{aligned}$$

But then complicated terms appear in (8) and the other equations and it is not clear how our approach can be adapted to this case. Therefore, we restrict our attention to the splitting described above. When the hierarchical basis are taken on $\mathscr{V}_{0,k}$ and $\mathscr{Q}_k$, the matrix $\mathscr{A}_k$ decomposes naturally in the block form

$$\mathscr{A}_k = \begin{pmatrix} \mathscr{A}_{cc}^{(k)} & \mathscr{A}_{cf}^{(k)} \\ \mathscr{A}_{fc}^{(k)} & \mathscr{A}_{ff}^{(k)} \end{pmatrix}.$$

The indices c and f are for coarse and fine. Of course, we have

$$\mathscr{A}_{cc}^{(k)} = \mathscr{A}_{k-1}.$$

We have similar decompositions for $\mathscr{B}_k$, $\mathscr{C}_k$ and $\mathscr{M}_k$. Also the unknown vectors split into

$$U_k = Y_{k-1} + Z_{k-1}, \qquad P_k = P_{k-1} + \Pi_k$$

and for the forcing and non-linear term we have

$$\begin{aligned} b(U_k) &= b_c(U_k) + b_f(U_k), \\ b_{\text{int}}(Y_{k-1}^n, Z_k^n) &= b_{\text{int},c}(Y_{k-1}^n, Z_k^n) + b_{\text{int},f}(Y_{k-1}^n, Z_k^n), \\ F_k &= F_{k-1} + G_k. \end{aligned} \tag{10}$$

With these notations, the matrix form of the time discretization of (8) is

$$\begin{cases} (\mathscr{M}^k_{cc} + \mathscr{A}^k_{cc})Y^{n+1}_{k-1} + (\mathscr{M}^k_{cf} + \mathscr{A}^k_{cf})Z^{n+1}_k + {}^t\mathscr{B}^k_{cc}P^{n+1}_{k-1} + {}^t\mathscr{B}^k_{fc}\Pi^{n+1}_k \\ \qquad = F_{k-1} + \mathscr{M}^k_{cc}Y^n_{k-1} + \mathscr{M}^k_{cf}Z^n_k - b_c(Y^n_{k-1}) - b_{\mathrm{int},c}(Y^n_{k-1}, Z^n_k), \\ \mathscr{B}^k_{cc}Y^{n+1}_{k-1} + \mathscr{B}^k_{cf}Z^{n+1}_k - \varepsilon\mathscr{C}^k_{cc}P^{n+1}_{k-1} - \varepsilon\mathscr{C}^k_{cf}\Pi^{n+1}_k = 0. \end{cases} \tag{11}$$

A similar form holds for (9). In (11), the unknown P^{n+1}_{k-1} can be eliminated yielding a similar form as (7):

$$\begin{aligned} \left(\mathscr{M}^k_{cc} + \mathscr{A}^k_{cc} + \frac{1}{\varepsilon}{}^t\mathscr{B}^k_{cc}(\mathscr{C}^k_{cc})^{-1}\mathscr{B}^k_{cc}\right)Y^{n+1}_{k-1} &= F_{k-1} - b_c(Y^n_{k-1}) + \mathscr{M}^k_{cc}Y^n_{k-1} \\ &\underline{-b_{\mathrm{int},c}(Y^n_{k-1}, Z^n_k) - (\mathscr{M}^k_{cf} + \mathscr{A}^k_{cf})Z^{n+1}_k} \\ &\underline{+\mathscr{M}^k_{cf}Z^n_k - {}^t\mathscr{B}^k_{fc}\Pi^{n+1}_k - \frac{1}{\varepsilon}{}^t\mathscr{B}^k_{cc}(\mathscr{C}^k_{cc})^{-1}\mathscr{B}^k_{cf}Z^{n+1}_k + {}^t\mathscr{B}^k_{cc}(\mathscr{C}^k_{cc})^{-1}\mathscr{C}^k_{cf}\Pi^{n+1}_k}. \end{aligned} \tag{12}$$

Note that if the underlined terms are omitted in (12), we obtain exactly the discrete formulation on the level $k-1$. (This corresponds to omitting the interaction term in (8).) Thus they appear as a correction; they are comparable to the Reynolds's stress tensor appearing in turbulence modelling. Our aim is to take those corrections into account without solving (9). We want to use similar ideas as Reference 4 where Fourier discretization on the 2D Navier–Stokes equations with periodic boundary conditions was used and as in Reference 8, where the 2D Burgers equations were discretized with finite elements. The key observation in these works was that the underlined interaction terms vary slowly and can be frozen on sufficiently long interval of time. A multilevel strategy based on V cycles was used.

This methodology has shown to be very effective for the 2D Navier–Stokes equations with periodic boundary conditions and has been generalized to 3D flows.

It has also been implemented successfully on the Burgers equations. However, in this case the length of the interval during which the interaction terms can be frozen was shorter and the way they were estimated was not totally satisfactory.

In the next section we show that, in our case, these problems can be overcome.

4. ANALYSIS OF THE INTERACTION TERMS

In this section, we generalize the multilevel algorithm used in Reference 8 to the Navier–Stokes equations discretized with finite elements. As mentioned above, the key is that the interaction terms vary slowly. We give a relation between the length of the intervals of time during which these terms can be frozen and their variations. Then we use a solution computed by the method described in section 2 to have an estimate of these lengths of time. It appears that the interaction terms can indeed be frozen and the algorithm can work. However, the length of the intervals are not very large and we propose a new approach based on the decomposition of the domain, it allows us to define naturally localized small scale components and we show that with this new decomposition the scales separate much better and the interaction terms can be frozen on long interval of time.

4.1. A two-level algorithm

The algorithm used in Reference 8 can easily be adapted to the case of a two level splitting of the unknowns. For clarity, we describe it on the semidiscretized equations (8), (9).

Starting at time t_0 with known $y_{k-1}(t_0)$ and $z_k(t_0)$, we determine two times $\tau_{NL}(t_0)$ and $\tau_L(t_0)$. We also have a fixed time τ_0 (which in practical is one or two time steps). The nonlinear interaction terms b_{int} will be kept at its value at time t_0 during the whole interval $[t_0, t_0 + \tau_{NL}(t_0)]$. The linear terms are frozen on the smaller interval $[t_0, t_1 - \tau_0]$, where $t_1 = t_0 + \tau_L(t_0) + \tau_0$. In other words, on $[t_0, t_1 - \tau_0]$, we approximate y_{k-1}, p_{k-1} by the solutions $\bar{y}_{k-1}, \bar{p}_{k-1}$ of

$$\left(\frac{\partial \bar{y}_{k-1}}{\partial t}, \tilde{y}_{k-1}\right) + \frac{1}{Re}((\bar{y}_{k-1}, \tilde{y}_{k-1})) + b(\bar{y}_{k-1}, \bar{y}_{k-1}, \tilde{y}_{k-1}) - (\bar{p}_{k-1}, \operatorname{div} \tilde{y}_{k-1})$$
$$= (f, \tilde{y}_{k-1}) - L(z_k(t_0), \pi_k(t_0), \tilde{y}_{k-1}) - b_{\text{int}}(y_{k-1}(t_0), z_k(t_0), \tilde{y}_{k-1}), \tag{13}$$
$$(\operatorname{div} \bar{y}_{k-1} + \varepsilon \bar{p}_{k-1}, \tilde{p}_{k-1}) = -(\operatorname{div} z_k(t_0) + \varepsilon \pi_k(t_0), \tilde{p}_{k-1}), \tag{14}$$

for any $\tilde{y}_{k-1}, \tilde{p}_{k-1}$ in $\mathscr{V}_{0,k-1} \times \mathscr{Q}_{k-1}$. Then we iterate the full system (5) on the interval $[t_1 - \tau_0, t_1]$ with initial data

$$u_k(t_1 - \tau_0) = \bar{y}_{k-1}(t_1 - \tau_0) + z_k(t_0) \tag{15}$$

and only the non-linear interaction terms being frozen, i.e. we approximate u_k, p_k by $\bar{u}_k$, $\bar{p}_k$ the solutions of

$$\begin{gathered}\left(\frac{\partial \bar{u}_k}{\partial t}, v_k\right) + \frac{1}{Re}((\bar{u}_k, v_k)) - (p_k, \operatorname{div} v_k) + b(\bar{y}_{k-1}, \bar{y}_{k-1}, v_k) \\ = (f, v_k) - b_{\text{int}}(y_{k-1}(t_0), z_k(t_0), v_k), \\ (\operatorname{div} \bar{u}_k, q_k) + \varepsilon(p_k, q_k) = O\end{gathered} \tag{16}$$

for any (v_k, q_k) in $\mathscr{V}_{0,k} \times \mathscr{Q}_k$ on the interval $[t_1 - \tau_0, t_1]$ with initial data (15).

In (16), $\bar{y}_{k-1}$ is the large-scale component of $\bar{u}_{k-1}$. Thus we have the values of $\bar{y}_{k-1}(t_1), \bar{z}_k(t_1)$ as well as of all the linear interaction terms.

We determine a new $\tau_L(t_1)$ and iterate this on the interval $[t_1, t_2]$ with $t_2 = t_1 + t_0 + \tau_L(t_1) + \tau_0$, i.e. we simulate (13), (14) on $[t_1, t_2 - \tau_0]$ with only the linear interaction terms $L(z_k(t_0), \pi_k(t_0), \tilde{y}_{k-1})$, $(\operatorname{div} z_k(t_0) + \varepsilon \pi_k(t_0), \tilde{p}_{k-1})$ being changed into $L(z_k(t_1), y_{k-1}(t_1), \tilde{y}_{k-1})$, $(\operatorname{div} z_k(t_1) + \varepsilon \pi_k(t_1), \tilde{p}_{k-1})$. The term b_{int} is still evaluated at time t_0. Then on $[t_2 - \tau_0, t_2]$, we simulate (16) with initial data

$$\bar{u}_k(t_2 - \tau_0) = \bar{y}_{k-1}(t_2 - \tau_0) + z_k(t_1).$$

We go on until time $t_0 + \tau_{NL}(t_0)$ is reached. Then all the interaction terms are reevaluated and a new τ_{NL} is determined.

4.2. *Estimate of* τ_{L} and τ_{NL}

This algorithm will be effective only if the times τ_L and τ_{NL} are sufficiently large. We derive a relation between τ_L, τ_{NL} and the variations of the interaction terms so that we will be able to estimate them. We still restrict our attention to the semidiscretized equations, the results being similar for the fully discretized equations.

Let $e_y = y_{k-1} - \bar{y}_{k-1}$, $e_{p_y} = p_{k-1} - \bar{p}_{k-1}$; substracting (13), (14) to (8) and taking e_y, e_{p_y} as test functions, we obtain on $[t_i, t_i + \tau_L(t_i))]$:

$$\begin{gathered}\frac{1}{2}\frac{\partial |e_y|^2}{\partial t} + \frac{1}{Re}\|e_y\|^2 - (e_{p_y}, \operatorname{div} e_y) = -b(y_{k-1}, y_{k-1}, e_y) + b(\bar{y}_{k-1}, \bar{y}_{k-1}, e_y) \\ - L(z_k, \pi_k, e_y) + L(z_k(t_i), \pi_k(t_i), e_y) - b_{\text{int}}(y_{k-1}, z_k, e_y) + b_{\text{int}}(y_{k-1}(t_0), z_k(t_0), e_y), \\ (\operatorname{div} e_y, e_{p_y}) = -\varepsilon |e_{p_y}|^2 - \varepsilon(\pi_k - \pi_k(t_i), e_y) - (\operatorname{div} z_k - \operatorname{div} z_k(t_i), e_{p_y}).\end{gathered} \tag{17}$$

We obtain after easy computations:

$$\begin{aligned}\frac{1}{2}\frac{\partial|e_y|^2}{\partial t}+\frac{1}{Re}\|e_y\|^2+\frac{\varepsilon}{2}|e_{p_y}|^2 &\leqslant |b(y_{k-1},y_{k-1},e_y)-b(\bar{y}_{k-1},\bar{y}_{k-1},e_y)|+|e_y|^2\\ &+\tfrac{1}{2}|b_{\mathrm{int}}(y_{k-1},z_k,\tilde{y}_{k-1})-b_{\mathrm{int}}(y_{k-1}(t_0),z_k(t_0),\tilde{y}_{k-1})|^2\\ &+\tfrac{1}{2}|L(z_k,\pi_k\tilde{y}_{k-1})-L(z_k(t_i),\pi_k(t_i),\tilde{y}_{k-1})|^2+\varepsilon|(\pi_k-\pi_k(t_i),\tilde{p}_{k-1})|^2\\ &+\frac{1}{\varepsilon}|(\mathrm{div}\,z_k-\mathrm{div}\,z_k(t_i),p_{k-1})|^2. \end{aligned}\tag{18}$$

We have used the notation $b_{\mathrm{int}}(y_{k-1},y_{k-1},\tilde{y}_{k-1})$ to denote the orthogonal projection of the non-linear interaction term on $\mathscr{V}_{k-1}$ and $L(z_k,\pi_k,\tilde{y}_{k-1})$, $(\pi_k,\tilde{p}_{k-1})$ or $(\mathrm{div}\,z_k,\tilde{p}_{k-1})$ have to be interpreted similarly.

The first term on the right-hand side of (18) is the same term that would appear when comparing y_{k-1} to the true solution of (5). The four last terms are introduced by our algorithm and represent additional discretization errors.

Let $\eta>0$ be a small number that would be fixed during the whole computation with the algorithm described above. This parameter is the maximum value of the perturbations that one wants to introduce. Over the interval $[t_0,t_0+\tau_{NL}(t_0)]$, the non-linear interaction term is frozen. The perturbation induced is represented by the third term and at the time $t_0+\tau_{NL}(t_0)$ is responsible of an accumulated perturbation which can be measured by

$$\begin{aligned}&\frac{1}{2}\int_{t_0}^{t_0+\tau_{NL}(t_0)}|b_{\mathrm{int}}(y_{k-1},z_k,\tilde{y}_{k-1})-b_{\mathrm{int}}(y_{k-1}(t_0),z_k(t_0),\tilde{y}_{k-1})|^2\,\mathrm{d}t\\ &\qquad\leqslant\frac{\tau_{NL}^3}{6}\sup_{[t_0,t_0+\tau_{NL}(t_0)]}\left|\frac{\partial}{\partial t}b_{\mathrm{int}}(y_{k-1},z_k,\tilde{y}_{k-1})\right|^2.\end{aligned}\tag{19}$$

Similarly, on each interval $[t_i,t_i+\tau_L(t_i)]$, we have a perturbation from the linear interaction term which is estimated by

$$\begin{aligned}&\frac{\tau_L(t_i)^3}{3}\left(\frac{1}{2}\sup_{[t_i,t_i+\tau_L(t_i)]}\left|\frac{\partial}{\partial t}L(z_k,\pi_k,\tilde{y}_{k-1})\right|^2\right.\\ &\qquad\left.+\varepsilon\sup_{[t_i,t_i+\tau_L(t_i)]}\left|\frac{\partial}{\partial t}(\pi_k,\tilde{p}_{k-1})\right|^2+\frac{1}{\varepsilon}\sup_{[t_i,t_i+\tau_L(t_i)]}\left|\frac{\partial}{\partial t}(\mathrm{div}\,z_k,\tilde{p}_{k-1})\right|^2\right).\end{aligned}\tag{20}$$

It seems natural to require that each of the right-hand sides of (19) and (20) is less than η^2 (recall they represent components of square of the error). These estimates can be derived in a heuristic way as follows. In (18), we consider that each terms acts separately on the error and that the error will be the sum of e_1, e_2, e_3 such that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|e_1|^2+\frac{1}{Re}\|e_1\|^2\leqslant|e_1|^2+|b(y_{k-1},y_{k-1},e_1)-b(\bar{y}_{k-1},\bar{y}_{k-1},e_1)|,\\ &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|e_2|^2+\frac{1}{Re}\|e_2\|^2\leqslant|e_2|^2+\frac{1}{2}|b_{\mathrm{int}}(y_{k-1},z_k,\tilde{y}_{k-1},)-b_{\mathrm{int}}(y_{k-1}(t_0),z_k(t_0),\tilde{y}_{k-1})|^2,\\ &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|e_3|^2+\frac{1}{Re}\|e_3\|^2\leqslant|e_3|^2+\frac{1}{2}|L(z_k,\pi_k,\tilde{y}_{k-1})-L(z_k(t_i),\pi_k(t_i),\tilde{y}_{k-1})|^2,\\ &\qquad+\varepsilon|(\pi_k-\pi_k(t_i),\tilde{p}_{k-1})|^2+\frac{1}{\varepsilon}|(\mathrm{div}\,z_k-\mathrm{div}\,z_k(t_i),\tilde{p}_{k-1})|^2.\end{aligned}$$

We do not want to consider e_1, since this component of the error is not due to our algorithm; e_2 and e_3 are easily estimated thanks to a Gronwall lemma and, neglecting the exponential term, we obtain (19), (20).

Also, when simulating the full system on each $[t_i - \tau_0, t_i]$, we have an error in the initial data which is

$$|z_k(t_i - \tau_0) - z_k(t_{i-1})| \leqslant \tau_L(t_i) \sup_{[t_i, t_i + \tau_L(t_i)]} \left| \frac{\partial z_k}{\partial t} \right|. \tag{21}$$

We require that this term is also less than η.

The time τ_0 is chosen so that this error is smoothed. Note that the small-scale components are damped very fast and it is legitimate to expect that τ_0 can be taken to be small. This time τ_0 can be interpreted as the time necessary for the small scales to adjust to the large scales.

4.3. A posteriori analysis

We now give estimates of the time τ_L and τ_{NL} by measuring the variations of the interaction terms on a computed solution.

We concentrate our analysis on three terms

1. $|(\partial z_k/\partial t, \tilde{y}_{k-1})|$ which behaves like $|1/Re((z_k, \tilde{y}_{k-1}))|$, $|(\pi_k, \operatorname{div} \tilde{y}_{k-1})|$, $|(\pi_k, \tilde{p}_{k-1})|$ or $|\partial z_k/\partial t|$.
2. $|(\operatorname{div} z_k, \tilde{p}_{k-1})|$ since $1/\varepsilon|\partial/\partial t(\operatorname{div} z_k, \tilde{p}_{k-1})|^2$ is the dominant term in the estimate of τ_L (see (20)).
3. The non-linear interaction term $|b_{\text{int}}(y_{k-1}, z_k, \tilde{y}_{k-1})|$.

Let us recall that $|\cdot|$ is the $L^2(\Omega)$ norm. During this computation, a term like $(\partial z_k/\partial t, \tilde{y}_{k-1})$ is represented by its scalar product with the elements of the hierarchical basis of $\mathscr{V}_{k-1}$ which are

$$\beta_i = \left(\frac{\partial}{\partial t} z_k, \Phi_{k-1,i} \right), \qquad i = 1, \ldots, n_{k-1}.$$

Let us write

$$\left(\frac{\partial}{\partial t} z_k, \tilde{y}_{k-1} \right) = \sum_{i=1}^{n_{k-1}} \alpha_i \Phi_{k-1,i},$$

then

$$\left(\frac{\partial}{\partial t} z_k, \Phi_{k-1,i} \right) = \sum_{j=1}^{n_{k-1}} \alpha_j (\Phi_{k-1,j}, \Phi_{k-1,i}) = \Delta t \left(\mathscr{M}_{k-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{n_{k-1}} \end{pmatrix} \right)_i.$$

Therefore

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n_{k-1}} \end{pmatrix} = \Delta t \mathscr{M}_{k-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{n_{k-1}} \end{pmatrix}$$

and

$$\left|\left(\frac{\partial}{\partial t}z_k, \tilde{y}_{k-1}\right)\right|^2 = \sum_{i,j=1}^{n_{k-1}} \alpha_i\alpha_j(\Phi_{k-1,i}, \Phi_{k-1,j})$$

$$= \Delta t(\alpha_1, \ldots, \alpha_{n_{k-1}})\mathscr{M}_{k-1}\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_{n_{k-1}}\end{pmatrix}$$

$$= (\Delta t)^{-1}(\beta_1, \ldots, \beta_{n_{k-1}})\mathscr{M}_{k-1}^{-1}\begin{pmatrix}\beta_1\\ \vdots \\ \beta_{n_{k-1}}\end{pmatrix}.$$

The inverse of $\mathscr{M}_{k-1}$ is computed by a mass lumping so that the evaluation of the $L^2(\Omega)$ norm is cheap. The other quantities are computed similarly.

The result below are for two splitting, for $k=2$ and 3.

4.3.1. Global analysis

On Figure 3, we have the $L^2(\Omega)$ norm of $(\partial z_k/\partial t, \tilde{y}_{k-1})$ and of its variation over one time step which is approximately

$$\Delta t\left|\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}z_k, \tilde{y}_{k-1}\right)\right|.$$

On Figures 4 and 5 the same quantities for $|(\text{div } z_k, \tilde{p}_{k-1})|$ and $|b_{\text{int}}(y_{k-1}, z_k, \tilde{y}_{k-1})|$. First, comparing with the top of Figure 3 where the $L^2(\Omega)$ norm of $(\partial y_{k-1}/\partial t, \tilde{y}_{k-1})$ is shown, we notice that the interaction terms are small but not sufficiently and they cannot be neglected.

Another remark is that in our computation the stationary flow establishes approximately at time $t=5$ and our analysis should consider only times less than 5 since we intend to get information on non-stationary flows.

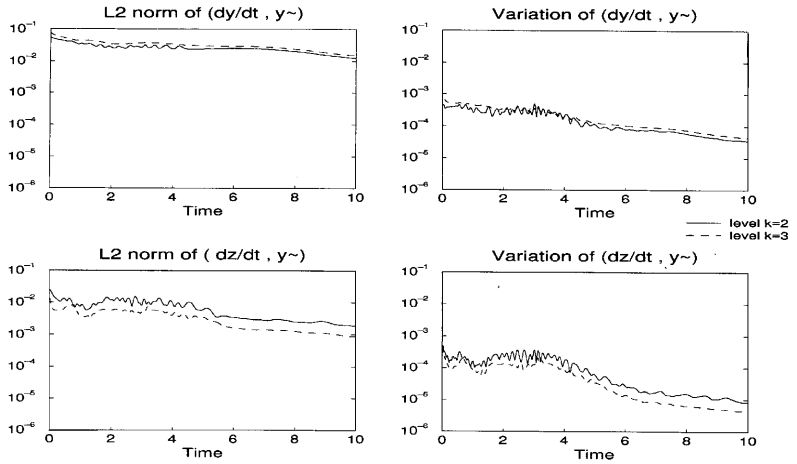

Figure 3. Global evolution of $\partial/\partial t$ terms

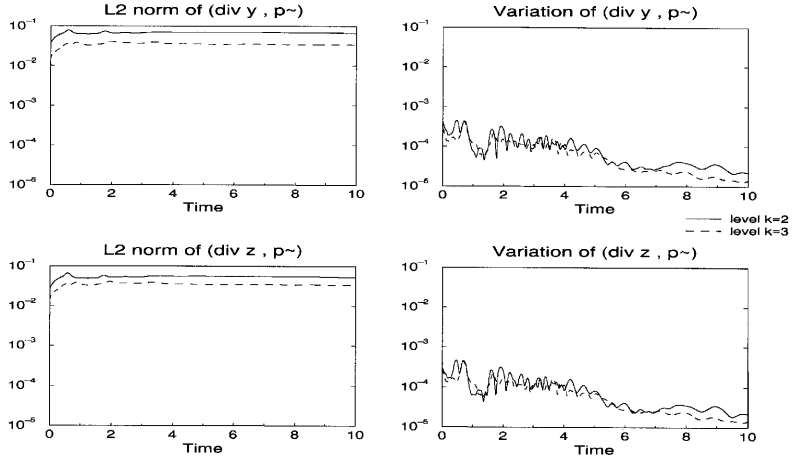

Figure 4. Global evolution of div terms

We choose the parameter $\eta = \Delta t = 2{\cdot}10^{-3}$, this is justified by the fact that we have an Euler scheme of order one in time which produces an error of this order.

From these results, we can take typical values of $|\partial b_{\text{int}}(y_{k-1}, z_k, \tilde{y}_{k-1})/\partial t|$, $|\partial(\text{div } z_k, \tilde{p}_{k-1})/\partial t|$ and $|\partial z_k/\partial t|$ to be $3 \times 10^{-4}(\Delta t)^{-1}$, $5 \times 10^{-4}(\Delta t)^{-1}$ and 5×10^{-2}. Thus (19), (20) and (21) will be approximately less than η^2 if

$$\tau_{NL} \leqslant 4 \times 10^{-2},$$
$$\tau_L \leqslant \min\{2 \times 10^{-3}, 2 \times 10^{-2}\}.$$

This indicates that the linear interaction term can be frozen on a few time steps (which equal 2×10^{-3} here) and the non-linear interaction term on about 40 time steps. We can foresee that the implementation of the algorithm, however, due to the above estimate, would not be very efficient.

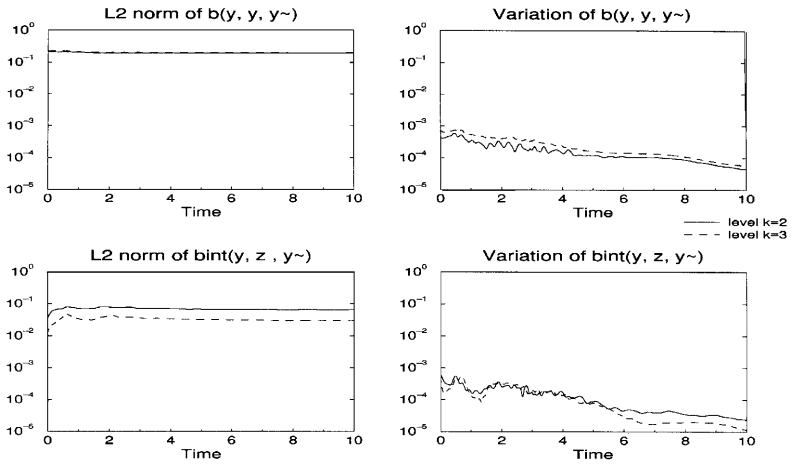

Figure 5. Global evolution of non-linear terms

Thus, we would like to increase these times, especially τ_L, and we propose a new approach below. Also, the quantity

$$\sup_{[t_0,t_0+\tau_{NL}(t_0)]}\left|\frac{\partial}{\partial t}b_{\text{int}}(y_{k-1},z_k,\tilde{y}_{k-1})\right|^2$$

is very expensive to estimate in practice. In Reference 15, it was found that it behaves like other simple quantities. In the equivalent of (19), it was replaced by

$$\sup_{[t_0,t_0+\tau]}\left|\frac{\partial z_k}{\partial t}\right|^2,$$

which is very cheap to estimate.

In Reference 4, a similar idea was used, but it was not clear how to justify this approximation in that case. In our case, we have not been able to find a simple quantity which could be used. We see below that this problem is overcome when the small scales are localized.

4.3.2. Local analysis

Since we are using finite elements, it seems natural to take advantage of the local type of this discretization. One way to do this is to decompose the domain of integration. We consider the simple case of a decomposition in four subdomains (Figure 6). Then the small components splits naturally in four terms, corresponding to a splitting of the basis into four parts, each corresponding to a subdomain. Also, all the interaction terms split into four components.

This approach could be coupled to multidomain decomposition technics and parallel computation. With this in mind we imagine a new possibility, which is to freeze the interaction term locally. The times τ_L and τ_{NL} would depend on the subdomain we consider. It is easy to see that they can be estimated by the same formula (19)–(21) as in the global case except that the $L^2(\Omega)$ norm has to be replaced by the $L^2(\Omega_i)$, $i=1,\ldots,4$, norm.

On Figures 7–8, 9–10 and 11– 12, we represent the three terms

$$\left|\frac{\partial}{\partial t}(z_k,\tilde{y}^i_{k-1})\right|_{L^2(\Omega_i)},\qquad |(\text{div } z_k,\tilde{y}^i_{k-1})|_{L^2(\Omega_i)},\qquad |b_{\text{int}}(y_{k-1},z_k,\tilde{y}^i_{k-1})|_{L^2(\Omega)}$$

for each subdomain and we see that their variations are much smaller but not at the same time, depending on the subdomain we consider. For instance, at time 2, the variations of $b_{\text{int}}(y_{k-1},z_k,\tilde{y}_{k-1})$ are about 10^{-5} in domain 1, 3×10^{-4} in domain 2, 5×10^{-6} in domain 3 and 10^{-5} in domain 4. If we use these numbers to estimate τ_{NL} in each domain, taking $\frac{1}{4}\eta$ instead of η, we estimate that the nonlinear interaction term can be frozen about 150 time steps in domain 1 and 4, 250 time steps in domain 3 and only 15 time steps in domain 2. At time $t=4$, on the contrary, τ_{NL} will be larger in

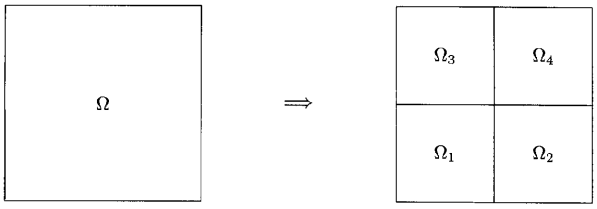

Figure 6. Subdomain decomposition

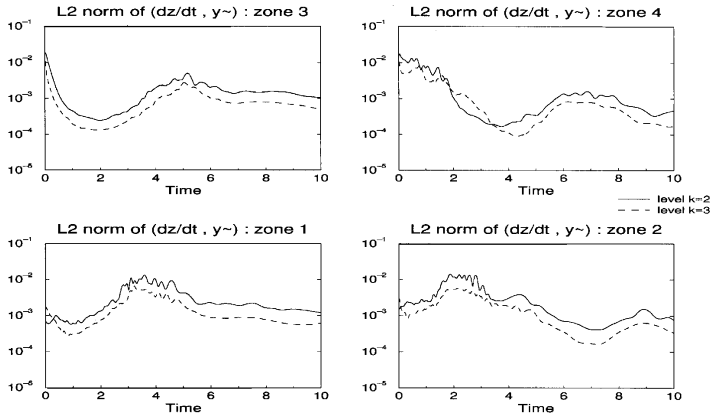

Figure 7. Local evolution of $(\partial z_k/\partial t, \tilde{y}_{k-1})$

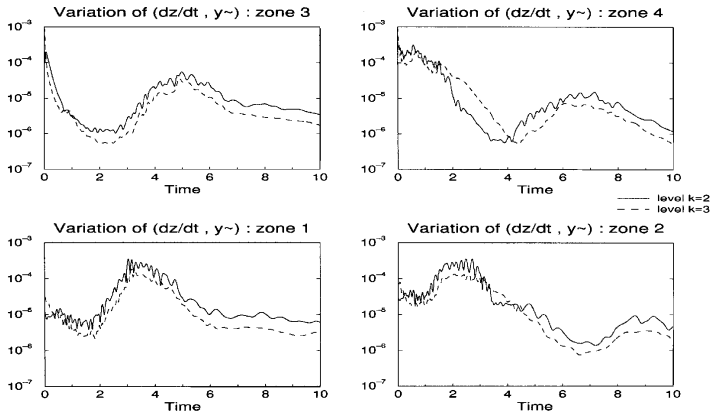

Figure 8. Local evolution of variation of $(\partial z_k/\partial t, \tilde{y}_{k-1})$

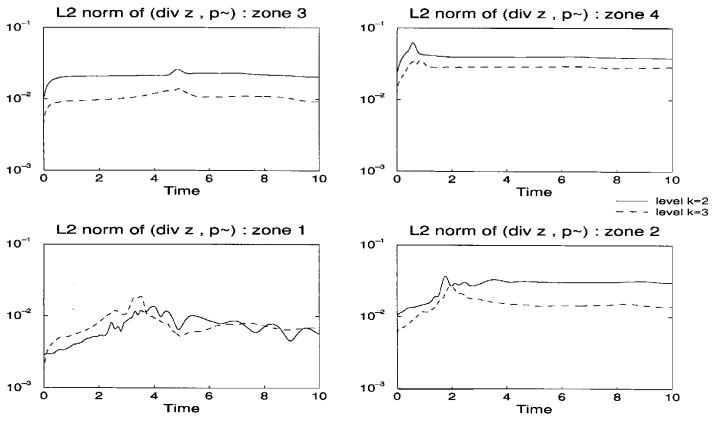

Figure 9. Local evolution of $(\text{div } z_k, \tilde{p}_{k-1})$

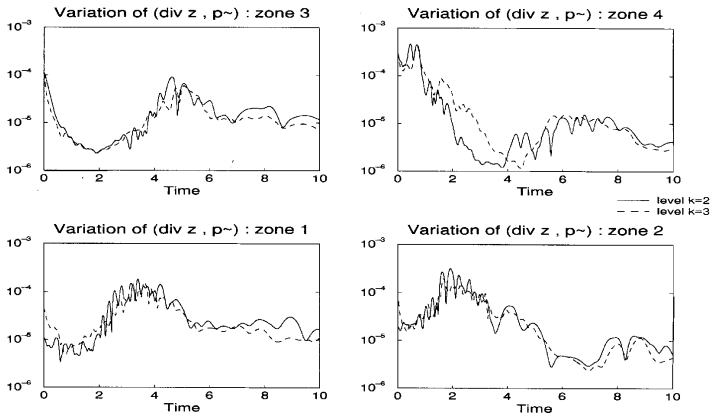

Figure 10. Local evolution of the variation of (div $z_k, \tilde{p}_{k-1}$)

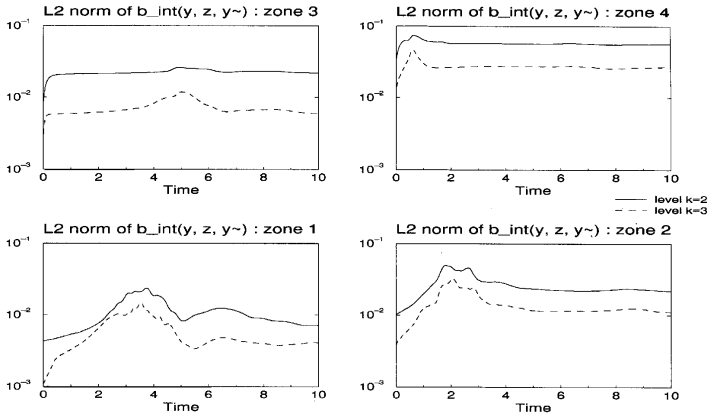

Figure 11. Local evolution of b_{int}

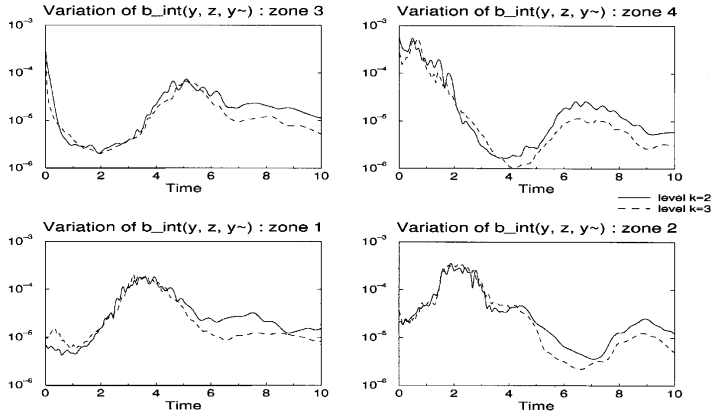

Figure 12. Local evolution of the variation of b_{int}

domain 4 and smaller in domain 1. We can do similar computation on τ_L and estimate its value, depending on the time and on the domain to vary between 1 and 30 time steps.

Thus τ_L and τ_{NL} can be very large when the small scale structures are localized and it seems that the algorithm described above—which can be easily extended to a multidomain computation—can be very effective.

Moreover, there is a clear correlation between the variations of $|\partial(z_k, \tilde{y}_{k-1})/\partial t|$ and $|\partial b_{\text{int}}(y_{k-1}, z_k, \tilde{y}_{k-1})/\partial t|$ in each subdomain, so that, after a correct scaling has been done we have a simple quantity which can be used to estimate τ_{NL} cheaply.

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